On ranking by first and last choosing

MICS: Algorithmic Decision Theory

Raymond Bisdorff

University of Luxembourg

January 20, 2021

Content

Sample performance tableau

Let $X = \{a_1, ..., a_7\}$ be seven potential decision actions evaluated on three cost criteria (g_1, g_4, g_5) of equi-significance 1/6 and two benefit criteria (g_2, g_3) of equi-signifiance 1/4. The given performance tableau is shown below.

Objectives	**Costs**			**Benefits**	
Criteria	$g_1(\downarrow)$	$g_4(\downarrow)$	$g_5(\downarrow)$	$g_2(\uparrow)$	$g_3(\uparrow)$
weights×12	2.0	2.0	2.0	3.0	3.0
indifference	3.41	4.91	-	-	2.32
preference	6.31	8.31	-	-	5.06
veto	60.17	67.75	-	-	48.24
a_1	22.49	36.84	7	8	43.44
a_2	16.18	19.21	2	8	19.35
a_3	29.41	54.43	3	4	33.37
a_4	82.66	86.96	8	6	48.50
a_5	47.77	82.27	7	7	81.61
a_6	32.50	16.56	6	8	34.06
a_7	35.91	27.52	2	1	50.82

Sample outranking relation

The resulting bipolar-valued outranking relation $\succsim$ is shown below.

Table: r-valued bipolar outranking relation

$r(\succsim) \times 12$	a_1	a_2	a_3	a_4	a_5	a_6	a_7
a_1	–	**0**	+8	+12	+6	+4	−2
a_2	+6	–	+6	+12	**0**	+6	+6
a_3	−8	−6	–	**0**	−12	+2	−2
a_4	−12	−12	**0**	–	−8	−12	**0**
a_5	−2	**0**	+12	+12	–	−6	**0**
a_6	+2	+4	+8	+12	+6	–	+2
a_7	+2	−2	+2	+6	**0**	+2	–

1. a_6 is a Condorcet winner,
2. a_2 is a weak Condorcet winner,
3. a_4 is a weak Condorcet looser.

Ranking by best-choosing and worst-rejecting – I

- Let X_1 be the set X of potential decision actions we wish to rank.
- While the remaining set X_i $(i = 1, 2, \ldots)$ of decision actions to be ranked is not empty, we extract from X_i the best (B_i), respectively worst (W_i) RUBIS choice recommendations and set $X_{i+1} = X_i - B_i$, respectively $X_{i+1} = X_i - W_i$.
- Both iterations determine, hence, two – usually slightly different – opposite weak rankings on X:
  1. a ranking-by-best-choosing and,
  2. a ranking-by-worst-rejecting.

Ranking by best-choosing and worst-rejecting – II

Ranking by recursively choosing:

```
>>> from transitiveDigraphs\
    import\
   RankingByBestChoosingDigraph
>>> rbbc =\
    RankingByBestChoosingDigraph(g)
>>> rbbc.showRankingByBestChoosing()
Ranking by recursively choosing
1st Best Choice ['a06']
  2nd Best Choice ['a02', 'a05']
    3rd Best Choice ['a07']
      4th Best Choice ['a01']
        5th Best Choice ['a03', 'a04']
```

Ranking by recursively rejecting:

```
>>> from transitiveDigraphs\
        import\
      RankingByLastChoosingDigraph
>>> rblc =\
    RankingByLastChoosingDigraph(g)
>>> rblc.showRankingByLastChoosing()
Ranking by recursively rejecting
1st Last Choice ['a03', 'a04']
  2nd Last Choice ['a05', 'a07']
   3rd Last Choice ['a06']
     4th Last Choice ['a01']
       5th Last Choice ['a02']
```

Notice the contrasted ranks of action a_5 (second best as well as second last) and action a_1 (fourth best as well as fourth last); indicating a lack of comparability, which becomes apparent in the disjunctive epistemic fusion R of both weak orderings.

Epistemic fusion of best-choosing and worst-rejecting

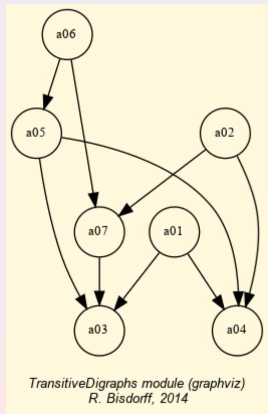
```
>>> fdg = FusionDigraph(rbbc,rblc); fdg.recodeValuation(-12,12)
>>> ranking = fdg.computeCopelandRanking()
>>> fdg.showRelationTable(Sorted=False,actionsSubset=ranking,\
                          ndigits=0,ReflexiveTerms=False)
```

Table: r-valued characteristics of the fusion digraph fdg

$r(x \succ y)$	a_6	a_2	a_1	a_5	a_7	a_4	a_3
a_2	0	0	0	0	+2	+12	+2
a_6	0	0	0	0	+2	+12	+6
a_1	0	0	0	0	0	+12	+8
a_5	−6	0	0	0	0	0	+12
a_7	−2	−2	0	0	0	0	+2
a_4	−12	−12	−12	−8	0	0	0
a_3	−2	−6	−8	−12	−2	0	0

Weak ranking by fusing best-choosing and worst-rejecting

```
>>> from transitiveDigraphs import\
              RankingByChoosingDigraph
>>> rbc = RankingByChoosingDigraph(g)
>>> rbc.showRankingByChoosing()
 Ranking by Choosing and Rejecting
  1st ranked ['a01', 'a02', 'a06'] (0.43)
    2nd ranked ['a05','a7'] (1.00)
    2nd last ranked ['a5','a07'] (1.00)
  1st last ranked ['a03', 'a04'] (0.62)
>>> rbc.exportGraphViz(fileName='rbc',\
                       direction='best')
 *- exporting a dot file for GraphViz tools -*
 Exporting to rbc.dot
 dot -Grankdir=TB -Tpng rbc.dot -o rbc.png
```



Bipolar characteristic function r – I

- $X = \{x, y, z, ...\}$ is a finite set of m decision alternatives;
- We define a binary relation R on X with the help of a bipolar characteristic function r taking values in the rational interval $[-1.0; 1.0]$.
- **Bipolar semantics**: For any pair $(x, y) \in X^2$,
  1. $r(x\,R\,y) = +1.0$ means $x\,R\,y$ *valid for sure*,
  2. $r(x\,R\,y) > 0.0$ means $x\,R\,y$ *more or less valid*,
  3. $r(x\,R\,y) = 0.0$ means both $x\,R\,y$ *and* $x \not R\, y$ *indeterminate*,
  4. $r(x\,R\,y) < 0.0$ means $x \not R\, y$ *more or less valid*,
  5. $r(x\,R\,y) = -1.0$ means $x \not R\, y$ *valid for sure*.

Bipolar characteristic function r – II

Boolean operations:
Let ϕ and ψ be two relational propositions.

1. negation: $r(\neg\phi) := -r(\phi)$.
2. disjunction: $r(\phi \vee \psi) := \max\big(r(\phi), r(\psi)\big)$,
3. conjunction: $r(\phi \wedge \psi) := \min\big(r(\phi), r(\psi)\big)$.
4. epistemic disjunction:

$$r(\phi \circledvee \psi) := \begin{cases} r(\phi \vee \psi) \text{ when } \big(r(\phi) \geqslant 0.0\big) \wedge \big(r(\psi) \geqslant 0.0\big) \\ r(\phi \wedge \psi) \text{ when} \big(r(\phi) \leqslant 0.0\big) \wedge \big(r(\psi) \leqslant 0.0\big) \\ 0.0 \quad \text{otherwise} \end{cases}$$

Weakly complete binary relations

Let R be an r-valued binary relation defined on X.

Definition

We say that R is weakly complete on X if, for all $(x, y) \in X^2$, **either** $r(x\,R\,y) \geqslant 0.0$ **or** $r(y\,R\,x) \geqslant 0.0$.

Examples

1. Marginal semi-orders (orders with discrimination thresholds) observed on each criterion,
2. Global weighted "*at least as performing as*" relations,
3. Outranking relations (polarized with considerable performance differences),
4. Fusion of (vague) weak or linear rankings,
5. Ranking-by-choosing results.

Universal properties

Let $\mathcal{R}$ denote the set of all possible weakly complete relations definable on X.

Property ($\mathcal{R}$-internal operations)

1. *The convex combination of any finite set of such weakly complete relations remains a weakly complete relation.*
2. *The disjunctive combination of any finite set of such weakly complete relations remains a weakly complete relation.*
3. *The epistemic-disjunctive (resp. -conjunctive) combination of any finite set of such weakly complete relations remains a weakly complete relation.*

Examples: Concordance of linear-, weak- or semi-orders, bipolar-valued outranking relations.

Useful properties

Definition (Coduality Principle)

We say that a binary relation $\succsim \in \mathcal{R}$ verifies the *coduality principle* when the converse of its negation equals its asymetric part : $\not\succsim^{-1} \equiv \succnsim$.

Let $\mathcal{R}^{cd}$ denote the set of all possible relations $R \in \mathcal{R}$ that verify the coduality principle.

Property

The convex and epistemic-disjunctive (resp. -conjunctive) combinations of a finite set of relations in $\mathcal{R}^{cd}$ verify again the coduality principle.

Examples: Marginal linear and weak rankings or orderings; orders with thresholds; bipolar-valued outranking relations; all, verify the coduality principle.

Pragmatic principles of the RUBIS choice

$\mathcal{P}_1$: Elimination for well motivated reasons:
Each eliminated alternative has to be outranked by (resp. is outranking) at least one alternative in the RUBIS choice (RC).

$\mathcal{P}_2$: Minimal size:
The RC must be as limited in cardinality as possible.

$\mathcal{P}_3$: Stable and efficient:
The RC must not contain a self-contained sub-RC.

$\mathcal{P}_4$: Effectively better (resp. worse):
The RC must not be ambiguous in the sense that it is not both a best choice as well as a worst choice recommendation.

$\mathcal{P}_5$: Maximally significant:
The RC is, of all potential best (resp. worst) choices, the one that is most significantly supported by the marginal "*at least as good as*" relations.

Qualifications of a choice in X

Let $\succsim$ be an r-valued outranking relation defined on X and let Y be a non empty subset of X, called a choice in X.

- Y is called outranking (resp. outranked) if for all non retained alternative x there exists an alternative y retained such that $r(y \succsim x) > 0.0$ (resp. $r(x \succsim y) > 0.0$).
- Y is called independent if for all $x \neq y$ in Y, we observe $r(x \succsim y) < 0.0$.
- Y is called weakly independent if for all $x \neq y$ in Y, we observe $r(x \succsim y) \leqslant 0.0$.
- Y is an outranking kernel (resp. outranked kernel) iff Y is an *outranking* (resp. *outranked*) and *independent* choice.
- Y is an outranking prekernel (resp. outranked prekernel) iff Y is an *outranking* (resp. *outranked*) and *weakly independent* choice.

Translating the pragmatic RUBIS principles in terms of choice qualifications

$\mathcal{P}_1$: Elimination for well motivated reasons.
The RC is an outranking choice (resp. outranked choice).

$\mathcal{P}_{2+3}$: Minimal and stable choice.
The RC is a prekernel.

$\mathcal{P}_4$: Effectivity.
The RC is a choice which is strictly more outranking than outranked (resp. strictly more outranked than outranking) .

$\mathcal{P}_5$: Maximal significance.
The RC is the most determined one in the set of potential outranking (resp. outranked) prekernels observed in a given r-valued strict outranking relation.

Properties of the RUBIS choice

Property (decisiveness)

Every r-valued strict outranking relation without chordless odd circuits admits at least one outranking and one outranked prekernel.

Definition

Let O and O' be two r-valued outranking relations defined on X.

1. We say that O' upgrades action $x \in X$, denoted $O^{x\uparrow}$, if $r(x\,O'\,y) \geqslant r(x\,O\,y)$, and $r(y\,O'\,x) \leqslant r(y\,O\,x)$, and $r(y\,O'\,z) \;=\; r(y\,O\,z)$ for all $y, z \in X - \{x\}$.
2. We say that O' downgrades action $x \in X$, denoted $O^{x\downarrow}$, if $r(y\,O'\,x) \geqslant r(y\,O\,x)$, and $r(x\,O'\,y) \leqslant r(x\,O\,y)$, and $r(y\,O'\,z) \;=\; r(y\,O\,z)$ for all $y, z \in X - \{x\}$.

Properties of the RUBIS choice

Let A be a subset of X. Let $RBC(O_{|A})$ (resp.$RBC(O'_{|A})$) be the RUBIS *best choice* wrt to O (resp. O') restricted to A; and, let $RWC(O_{|A})$ (resp.$RWC(O'_{|A})$) be the RUBIS *worst choice* wrt to O (resp. O') restricted to A.

Property

1. $O_{|A} = O'_{|A} \;\Rightarrow\; RBC(O_{|A}) = RBC(O'_{|A})$ *(RBC local)*,
2. $O_{|A} = O'_{|A} \;\Rightarrow\; RWC(O_{|A}) = RWC(O'_{|A})$ *(RWC local)*,
3. $x \in RBC(O_{|A}) \;\Rightarrow\; x \in RBC(O^{x\uparrow}_{|A})$ *(RBC weakly monotonic)*,
4. $x \in RWC(O_{|A}) \;\Rightarrow\; x \in RWC(O^{x\downarrow}_{|A})$ *(RWC weakly monotonic)*.

Ranking-by-Choosing Algorithm

1. Let X_1 be the set X of potential decision actions we wish to rank on the basis of a given outranking relation O.
2. While the remaining set X_i $(i = 1, 2, \ldots)$ of decision actions to be ranked is not empty, we extract from X_i the *best* (B_i), respectively *worst* (W_i), RUBIS choice recommendation and set $X_{i+1} = X_i - B_i$, respectively $X_{i+1} = X_i - W_i$.
3. Both independent iterations determine, hence, two – usually slightly different – opposite *weak rankings* on X: a ranking by-best-choosing – and a ranking by-last-choosing.
4. We fuse both weak rankings with the epistemic disjunction operator $(\otimes)$ to make apparent a *weakly complete* ranking relation $\succnsim_O$ on X.

Transitive $\succsim$-closure

Definition

We call a ranking procedure weakly transitive if the ranking procedure renders a (partial) strict ranking $\succnsim$ on X from a given r-valued outranking relation $\succsim$ such that for all $x, y, z \in X$: $r(x \succnsim y) \geqslant 0$ and $r(y \succnsim z) \geqslant 0$ imply $r(x \succnsim z) \geqslant 0$.

Property

Both the RUBIS *ranking-by-best-choosing, as well as the* RUBIS *ranking-by-last-choosing procedures, are weakly transitive ranking procedures.*

Corollary

i) The fusion of the ranking by RUBIS *best choice and the converse of the ranking by* RUBIS *last choice of a given r-valued outranking relation $\succsim$ is a weakly transitive ranking procedure.*
ii) The RUBIS *ranking-by-choosing represents a weakly transitive closure of the outranking relation $\succsim$.*

Weak monotinicity

Definition

We call a ranking procedure weakly monotonic if for all $x, y \in X$: $(x \succnsim y) \;\Rightarrow\; (x \succnsim^{x\uparrow} y)$ and $(y \succnsim x) \;\Rightarrow\; (y \succnsim^{x\downarrow} x)$,

Property

The ranking by RUBIS *best choice and the ranking by* RUBIS *last choice are, both, weakly monotonic ranking procedures.*

Corollary

The ranking-by-choosing, resulting from the fusion of the ranking by RUBIS *best choice and the converse of the ranking by* RUBIS *last choice, is hence a weakly monotonic procedure.*

Condorcet consistency

Definition

We call a ranking procedure Condorcet-consistent if the ranking procedure renders the same linear (resp. weak) ranking $\succnsim$ on X which is, the case given, modelled by the strict majority cut of the codual of a given $\succsim$ relation.

Property

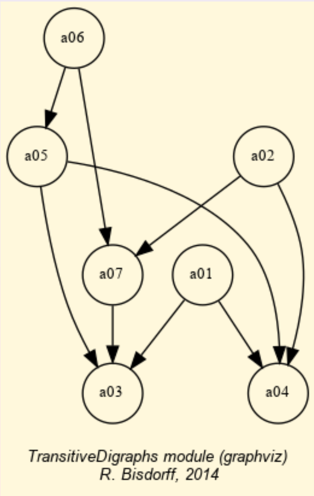
Both the RUBIS *ranking-by-best-choosing, as well as the* RUBIS *ranking-by-worst-choosing procedures, are Condorcet consistent.*

Corollary

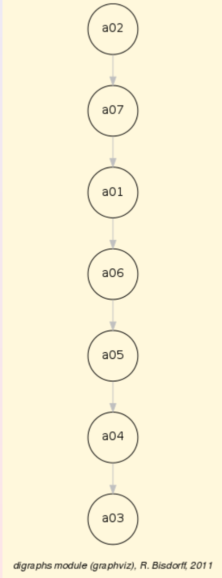
The fusion of the ranking by RUBIS *best choice and the ranking by* RUBIS *worst choice of a given r-valued outranking relation O is, hence, also Condorcet consistent.*

Introductory example

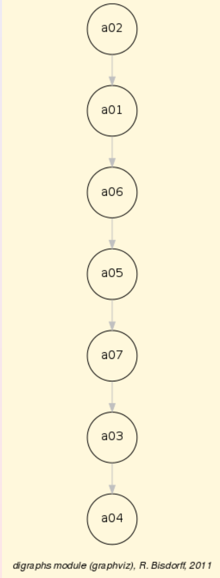
Comparing ranking-by-choosing result with Tideman's and Kohler's:



0.95%



0.91%



0.87%

Sample performance tableau

Let $X = \{a_1, ..., a_7\}$ be seven potential decision actions evaluated on three cost criteria (g_1, g_4, g_5) of equi-significance 1/6 and two benefit criteria (g_2, g_3) of equi-signifiance 1/4. The given performance tableau is shown below.

Objectives	Costs			Benefits	
Criteria	$g_1(\downarrow)$	$g_4(\downarrow)$	$g_5(\downarrow)$	$g_2(\uparrow)$	$g_3(\uparrow)$
weights×12	2.0	2.0	2.0	3.0	3.0
indifference	3.41	4.91	-	-	2.32
preference	6.31	8.31	-	-	5.06
veto	60.17	67.75	-	-	48.24
a_1	22.49	36.84	7	8	43.44
a_2	16.18	19.21	2	8	19.35
a_3	29.41	54.43	3	4	33.37
a_4	82.66	86.96	8	6	48.50
a_5	47.77	82.27	7	7	81.61
a_6	32.50	16.56	6	8	34.06
a_7	35.91	27.52	2	1	50.82

Quality of ranking result

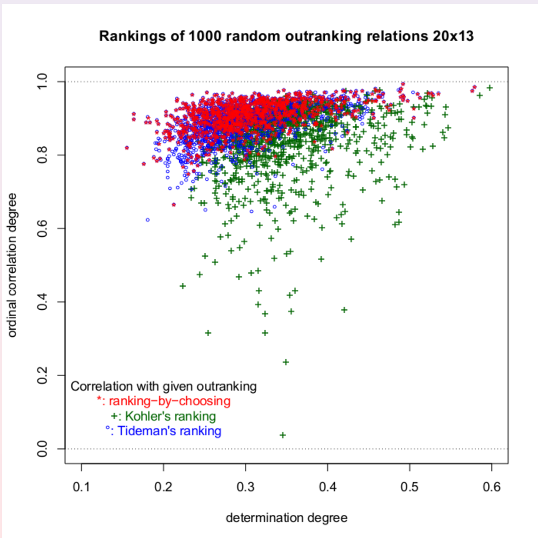
Comparing rankings of a sample of 1000 random r-valued outranking relations defined on 20 actions and evaluated on 13 criteria obtained with RUBIS **ranking-by-choosing**, **Kohler's**, and **Tideman's** (ranked pairs) procedure.

Mean extended Kendall τ correlations with r-valued outranking relation:

Ranking-by-choosing: $+.906$

Tideman's ranking: $+.875$

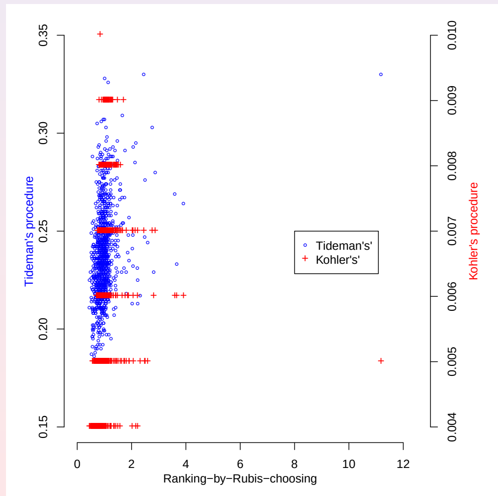
Kohler's ranking: $+.835$



Scalability of ranking procedures

Ranking execution times (in sec.) for 1000 random 20x13 outrankings:

- **Kohler**'s procedure on the right y-axis (less than **1/100** sec.),
- **Tideman**'s procedure on the left y-axis (less than **1/3** sec.),
- the RUBIS **ranking-by-choosing** procedure on the x-axis (mostly less than **2** sec.). But, heavy right tail (up to **11** sec. !).



Bibliography

[1] D. Bouyssou, *Monotonicity of 'ranking by choosing'; A progress report*. Social Choice Welfare (2004) 23: 249-273.

[2] R. Bisdorff, M. Pirlot and M. Roubens, *Choices and kernels from bipolar valued digraphs*. European Journal of Operational Research, 175 (2006) 155-170.

[3] R. Bisdorff, P. Meyer and M. Roubens, Rubis*: a bipolar-valued outranking method for the choice problem*. 4OR, A Quarterly Journal of Operations Research, Springer-Verlag, Volume 6 Number 2 (2008) 143-165.

[4] R. Bisdorff, *On measuring and testing the ordinal correlation between bipolar outranking relations*. In Proceedings of DA2PL'2012 - *From Multiple Criteria Decision Aid to Preference Learning*, University of Mons (2012) 91-100.

[5] R. Bisdorff, *On polarizing outranking relations with large performance differences*. Journal of Multi-Criteria Decision Analysis, Wiley, Number 20 (2013) DOI: 10.1002/mcda.1472 3-12.