Algorithmic Decision Theory

Lecture 3: On consensual social ranking

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20 janvier 2021

Content

Definition of the ranking problem

A ranking rule is a procedure which aggregates marginal, ie individual voters, experts or criteria based, rankings into a global *consensus ranking* which combines the available preferential information *best* from the marginal viewpoints.

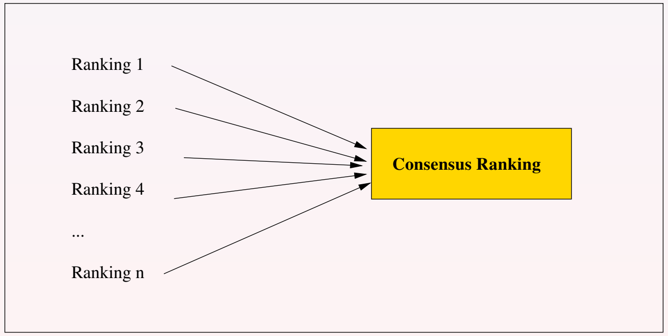


FIGURE – 1. Computing a consensual ranking

Example (A first example : Borda's average ranks)

```
>>> from votingProfiles import *
>>> v = LinearVotingProfile('example1')
>>> v.showLinearBallots()
  voters     marginal candidate's
 (weight)           rankings
  v1(8):   ['a', 'c', 'b', 'e', 'd']
  v2(7):   ['e', 'b', 'c', 'd', 'a']
  v3(4):   ['d', 'c', 'b', 'e', 'a']
  v4(4):   ['b', 'd', 'e', 'c', 'a']
  v5(2):   ['c', 'd', 'b', 'e', 'a']
 # voters: 25
>>> v.showRankAnalysisTable()
--------  Borda rank analysis tableau ------------------
 candi- |     candidate x rank       |     Borda
 dates  |  1     2     3     4    5    | score   average
--------|-----------------------------------------------
  'b'   |  4     7    14     0    0    |  60     2.40
  'c'   |  2    12     7     4    0    |  63     2.52
  'e'   |  7     0     4    14    0    |  75     3.00
  'd'   |  4     6     0     7    8    |  84     3.36
  'a'   |  8     0     0     0   17    |  93     3.72
```

Linear Rankings

- A linear ranking $R = [a_1, a_2, ..., a_n]$ is a list of n objects (a set X of candidates or decision alternatives) where the indexes $1 \leqslant i < j \leqslant n$ represent a complete preferential '*a_i better than a_j*' relation without ties ($a_i > a_j$). The reversed list is called a *linear order*.
- A linear ranking R may be modelled with the help of a *bipolar characteristic function* $r(a_i > a_j) \in \{-1, 0, 1\}$ where :

$$r(a_i > a_j) = \begin{cases} +1 \text{ if } i < j, \\ -1 \text{ if } i > j, \\ 0 \quad \text{otherwise.} \end{cases}$$

- Notice that reversing a ranking R is achieved by negation : $r(a_i \not> a_j) \; = \; -r(a_i > a_j)$ which characterizes the corresponding linear order.

Properties of linear rankings

A *linear ranking* $R = [a_1, a_2, ..., a_n]$ is

- a *transitive* relation, $\forall i, j, k = 1..n$:

$$\big[\, (r(a_i > a_j) = +1\,) \wedge (\, r(a_j > a_k) = +1\,)\big] \;\Rightarrow\; (r(a_i > a_k) = +1);$$

- a *complete* relation, $\forall i \neq j$:

$$r((a_i > a_j) \;\vee\; (a_i > a_j)) \;=\; \max\big(r(x_i > x_j), r(x_j > x_i)\big) \;=\; +1\,;$$

- an *irreflexive* relation, $\forall i$:

$$r(a_i > a_i) = 0 \quad /\text{* We ignore the reflexive relations */}.$$

- A ranking with ties –a collection of ordered equivalence classes– is called a weak ranking ; its *converse* is called a *preorder*, and its *negation* is called a weak order.

Majority margins

The majority margin $M(x,y)$ counts the *net advantage* of a candidate x over a candidate y. With k voters :

$$\begin{aligned} M(x,y) \;&=\; \sum_{k=1}^{n}\big(r(x >_k y)\big) + \sum_{k=1}^{n}\big(r(y \not>_k x)\big) \\ &=\; \sum_{k=1}^{n}\big[\,r(x >_k y) \;-\; r(y >_k x)\,\big] \end{aligned}$$

If the profile u consist of complete linear rankings, then :

$$M(x,x) \;=\; 0 \quad \text{and} \quad M(x,y) + M(y,x) \;=\; 0.$$

In this case, indeed :

$$\sum_{k=1}^{n}\big(r(x >_k y)\big) \;=\; n - \sum_{k=1}^{n}\big(r(y >_k x)\big)$$

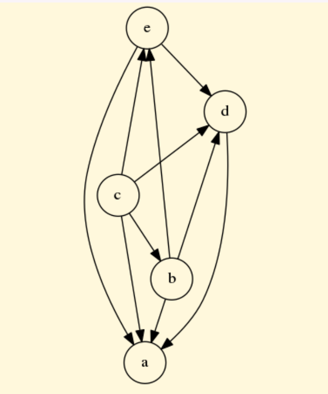
Example (Computing majority margins)

```
>>> v
Instance class    : LinearVotingProfile
Instance name     : example1
# Candidates: 5, # Voters: 5
>>> v.showLinearBallots()
  coalition     marginal candidate's
  (weight)            rankings
    v1(8):    ['a', 'c', 'b', 'e', 'd']
    v2(7):    ['e', 'b', 'c', 'd', 'a']
    v3(4):    ['d', 'c', 'b', 'e', 'a']
    v4(4):    ['b', 'd', 'e', 'c', 'a']
    v5(2):    ['c', 'd', 'b', 'e', 'a']
 Total number of voters: 25
>>> from votingProfiles import CondorcetDigraph
>>> cd = CondorcetDigraph(v)
>>> cd.showMajorityMargins()
* ---- Relation Table -----
 M(x,y)  |  'a'  'b'  'c'  'd'  'e'
---------|-------------------------------------
  'a'    |  0    -9   -9   -9   -9
  'b'    |  9     0   -3   13   11
  'c'    |  9     3    0    9    3
  'd'    |  9   -13   -9    0   -5
  'e'    |  9 - 11    -3    5    0
Valuation domain: [-25;+25]
```

The majority relation $C(x,y)$ checks if a majority margin $M(x,y)$ is *positive*, ie if there is a majority of rankings which rank candidate x before candidate y :

$$C(x,y) \;=\; \begin{cases} +1 & \text{if } M(x,y) > 0 \\ -1 & \text{if } M(x,y) < 0 \\ 0 & \text{otherwise.} \end{cases}$$

```
>>> cdp = PolarisedDigraph(cd,level=0,\
     StrictCut=True,KeepValues=False)
>>> cdp.recodeValuation(-1,1)
>>> cdp.showRelationTable(ndigits=0)
* ---- Relation Table -----
 C(x,y)  |  'a'   'b'   'c'   'd'   'e'
---------|-----------------------------
   'a'   |   0    -1    -1    -1    -1
   'b'   |   1     0    -1     1     1
   'c'   |   1     1     0     1     1
   'd'   |   1    -1    -1     0    -1
   'e'   |   1    -1    -1     1     0
>>> cdp.exportGraphViz()
```



Rubis Python Server (graphviz), R. Bisdorff, 2008

On ranking from different opinions

1. On ranking from different opinions
   - Definition of the ranking problem
   - Linear Rankings
   - Majority margins
2. Types of ranking rules
   - Borda type rules
   - Condorcet : Ranking-by-choosing rules
   - Condorcet : Ranking-by-scoring rules
3. A classification of ranking rules
   - Condorcet-consistency
   - M-ordinality and M-invariance
   - Which ranking rule should we use ?

Ranking rule

- A *profile* $u = \{R_1, R_2, ..., R_q\}$ is a list of q linear rankings.
- This profile u is the input of a ranking rule : $u \rightarrow f(u)$.
- The output of a ranking rule can be :
  - one (*SLR*) or several (*MLR*) *linear* rankings ;
  - one (*SWR*) or several (*MWR*) *weak* rankings (with ties).
- We present hereafter three types of ranking rules :
  1. *Rank analysis* based ranking-by-scoring rules (Borda type) ;
  2. *Pairwise majority margins* based rules (Condorcet type) :
     - 2.1 Ranking-by-choosing rules ;
     - 2.2 Ranking-by-scoring rules.

Borda's candidate-to-rank matrix

The *candidate-to-rank matrix* Q_{ij} counts the number of times the candidate a_i is ranked at position j.

$$Q_{ij} \;=\; \{\, \# \text{ rankings} : a_i \text{ is ranked at the } j \text{ th position}\}$$

```
Borda rank analysis tableau
voter's | marginal | candi-  |          Q_ij
weight  | ranking  | dates   |  1     2     3     4     5
--------|-----------------------------------------------
   8    |  acbed   |  'a'    |  8     0     0     0    17
   7    |  ebcda   |  'b'    |  4     7    14     0     0
   4    |  dcbea   |  'c'    |  2    12     7     4     0
   4    |  bdeca   |  'd'    |  4     6     0     7     8
   2    |  cdbea   |  'e'    |  7     0     4    14     0
```

Borda's rule

- A Borda score B is computed for each candidate a_i as follows :

$$B(a_i) \; = \; \sum_{j=1}^{n} (Q_{ij} \times j)$$

  The candidates are ranked from the lowest to the largest according to the Borda scores (to be mimized).
- A generalization of the Borda rule is to use any set of weights representing the ranks. Let $w_1 < w_2 < ... < w_n$ be increasing weights of the ranks. Then the Borda scores B are defined as follows :

$$B(a_i) \; = \; \sum_{j=1}^{n} (Q_{ij} \times w_j)$$

- The Borda ranking $\succeq_B$ is the weak ranking defined as follows :

$$\forall x, y \in X, \quad (x, y) \in \succeq_B \quad \Leftrightarrow \quad b_x \leqslant b_y.$$

Example (Borda's weighted scores)

Borda rank analysis tableau

candi- dates	candidate x rank 1	2	3	4	5	Borda scores w1	w2
'b'	4	7	14	0	0	60	88
'c'	2	12	7	4	0	63	85
'e'	7	0	4	14	0	75	111
'd'	4	6	0	7	8	84	122
'a'	8	0	0	0	17	93	144
w1	1	2	3	4	5		
w2	1	2	5	6	8		

We observe two different rankings : R_{w1} : *bceda* and R_{w2} : *cbeda*, depending hence on the actual rank weights. Notice that the original Borda ranking R_{w1} is not consistent with the majority relation, which is R_{w2}. Given the ordinal nature of the input data, there is no information on how to assign weights to the ranks.

Generalized ranks-based rules

Definition (Borda type Rules, *SWR*/candidate×rank analysis)

Let r_{ik}, $i = 1..n, k = 1..q$ be the rank of candidate a_i in ranking R_k, and $w_1, w_2, ..., w_q$ be a set of given rank weights. We may rank :

1. according to the average weighted rank :

$$B(a_i) \; = \; \frac{1}{q} \sum_{k=1}^{q} (r_{ik} \times w_k)$$

2. according to the weighted median rank :

$$B(a_i) \; = \; \text{median}\big[(r_{i1} \times w_1), (r_{i2} \times w_2), ..., (r_{iq} \times w_q)\big)$$

3. by minimizing a given distance function (Cook & Seiford).

Condorcet : Ranking-by-choosing Rules

Definition (Kohler's Rule, *MLR*/majority margins $M(x, y)$)

Optimistic sequential maximin rule. At step r (where r goes from 1 to n) :

1. Compute for each candidate x the smallest $M(x, y)$ $(x \neq y)$;
2. Select the candidate for which this minimum is maximal. If there are ties select in lexicograpic order ;
3. Put the selected candidate at rank r in the final ranking ;
4. Delete the row and the column corresponding to the selected candidate and restart from (1).

Example (Kohler's ranking rule)

4	a b c d
4	d c a b
4	c a b d
5	d b c a
1	c b d a
3	b c a d
4	d a b c
2	c d a b
2	b a c d
1	a c d b

	a	b	c	d	min
a		8	-8	-2	-8
b	-8		6	-2	-8
c	8	-6		4	-6
d	2	2	-4		-4

	a	b	c	min
a		8	-8	-8
b	-8		6	-8
c	8	-6		-6

	a	b	min
a		8	8
b	-8		-8

Kohler's ranking : d > c > a > b

FIGURE – 2. Source : Cl. Lamboray

Ranking-by-choosing Rules – continue

Definition (Arrow & Raynaud's Rule, *MLR*/majority margins $M(x, y)$)

Pessimistic (prudent) sequential minmax rule. At step r (where r goes from 1 to n) :

1. Compute for each candidate x the largest $M(x, y)$ $(x \neq y)$;
2. Select the candidate for which this maximum is minimal. If there are ties select the candidates in lexicographic order ;
3. Put the selected candidate at rank $n - r + 1$ in the final ranking ;
4. Delete the row and the column corresponding to the selected candidate and restart from (1).

Ranking-by-choosing Rules – continue

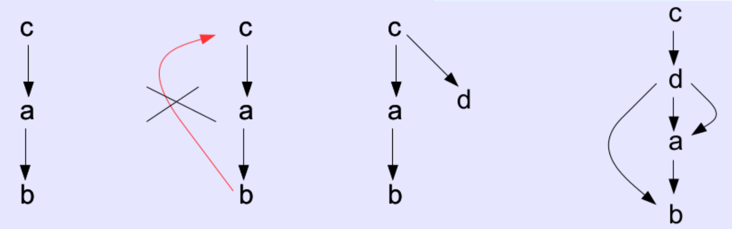
Definition (Ranked Pairs' Rule, *MLR*/majority margins $M(x, y)$)

1. Rank in decreasing order the ordered pairs (x, y) of candidates according to their majority margin $M(x, y)$.
2. Take any linear ranking compatible with this weak order.
3. Consider the pairs (x, y) in that order and do the following :
   - 3.1 If the considered pair creates a cycle with the already blocked pairs, skip this pair ;
   - 3.2 If the considered pair does not create a cycle with the already blocked pairs, block this pair.

Example (Ranked Pairs rule)

	a	b	c	d
a		8	-8	-2
b	-8		6	-2
c	8	-6		4
d	2	2	-4	

(c,a) (a,b)	8
(b,c)	6
(c,d)	4
(d,a) (d,b)	2
(a,d) (b,d)	-2
(d,c)	-4
(c,b)	-6
(a,c) (b,a)	-8



Ranked Pairs ranking : c > d > a > b

FIGURE – 3. Source : Cl. Lamboray

Example (Condorcet : Ranking-by-choosing)

```
>>> from linearOrders import *
>>> ko = KohlerOrder(cd)
>>> ko.kohlerRanking
['c', 'b', 'e', 'd', 'a']
>>> cdcd = ~(-cd)  # codual of cd
>>> ar = KohlerOrder(cdcd) # Arrow-Raynaud rule
>>> ar.kohlerRanking
['c', 'b', 'e', 'd', 'a']
>>> rp = RankedPairsOrder(cd)
>>> rp.rankedPairsRanking
['c', 'b', 'e', 'd', 'a']
```

Kohler's, Arrow&Raynaud's and the RankedPairs rule all result in the same unique linear ranking : '*cbeda*', which corresponds to the majority relation C.

Condorcet : Ranking-by-scoring rules

Definition (NetFlows Rule, *MWR*/majority margins $M(x, y)$)

- The idea is that the more a given candidate beats other candidates the better it is.
- Similarly, the more other candidates beat a given candidate, the lower this candidate should be ranked.
- The NetFlows score n_x of candidate x is defined as follows :

$$n_x = \sum_y \big[M(x,y) - M(y,x) \big].^1$$

- The NetFlows ranking $\succeq_N$ is the weak ranking defined as follows : $\forall x, y \in X, \quad (x,y) \in \succeq_N \quad \Leftrightarrow \quad n_x \geqslant n_y$.

1. Notice that in the case of linear profiles, we may drop the $-M(y,x)$ term due to the zero sum property.

Condorcet : Ranking-by-scoring rules

Definition (Copeland's Rule, *MWR*/majority relation $C(x, y)$)

- The idea is that the more a given candidate beats other candidates at majority the better it should be ranked.
- Similarly, the more other candidates beat a given candidate at majority, the lower this candidate should be ranked.
- The Copeland score c_x of candidate x is defined as follows :

$$\begin{aligned} c_x &= \#\{y \neq x \in X : M(x,y) > 0\} \\ &\quad - \#\{y \neq x \in X : M(y,x) > 0\} \\ &= \sum_y \big(C(x,y) - C(y,x) \big). \end{aligned}$$

- The Copeland ranking $\succeq_C$ is the weak ranking defined as follows : $\forall x, y \in X, \quad (x,y) \in \succeq_C \quad \Leftrightarrow \quad c_x \geqslant c_y$.

Condorcet : Ranking-by-scoring rules

Definition (Kemeny's Rule, *MLR*/majority margins $M(x, y)$)

- The idea is finding a compromise ranking R that minimizes the distance to the q marginal linear rankings of the voting profile according to the symmetric difference measure : δ. If R_1 and R_2 are two relations, $\delta(R_1, R_2) = |R_1 \oplus R_2| / 2$.
- The Kemeny ranking, also called *median* ranking, R^* is a solution of the following optimization problem : :

$$\text{minarg}_R\ \delta(M, R) \equiv \text{maxarg}_R \sum_{(x,y) \in R} \Big[M(x,y) \times r(x\, R\, y) \Big]$$

such that R is a linear ranking.

- The distance $\delta(M, R^*)$ is called the Kemeny index of a preference profile. Computing the Kemeny index is an NP-complete problem and Kemeny rankings are generally not unique.

Condorcet : Ranking-by-scoring rules

Definition (Slater's Rule, *MLR*/majority relation $C(x,y)$)

- The idea is to select a ranking that is closest according to the symmetric difference distance δ to the Condorcet digraph's polarized relation $M_{>0}$.
- The Slater ranking R^* is a solution of the following optimization problem :

$$\text{minarg}_R\ \delta(M_{>0}, R) \quad \equiv \quad \text{maxarg}_R \sum_{(x,y)\in R} \Big[C(x,y) \times r(x\,\mathrm{R}\,y)\Big]$$

such that R is a linear ranking.

- The distance $\delta(R^*, M_{>0})$ is called the Slater index of a preference profile. Computing the Slater index of a profile is an NP-hard problem and Slater rankings are even less unique than Kemeny rankings.

Example (Condorcet : ranking-by-scoring)

```
* ---- Majority margins -----
  Mxy  |   'a'    'b'    'c'    'd'    'e'
   'a' |    0     -9     -9     -9     -9
   'b' |    9      0     -3     13     11
   'c' |    9      3      0      9      3
   'd' |    9    -13     -9      0     -5
   'e' |    9    -11     -3      5      0
>>> cd.computeNetFlowsRanking(Debug=True)
OrderedDict([('b',60),('c',48),('e',0),('d',-36),('a',-72.0)])
['b', 'c', 'e', 'd', 'a']
>>> cd.computeCopelandRanking(Debug=True)
OrderedDict([('c', 4), ('b', 2), ('e', 0), ('d', -2), ('a', -4)])
['c', 'b', 'e', 'd', 'a']
>>> from linearOrders import KemenyOrder
>>> ke = KemenyOrder(cd); ke.maximalRankings
[['c', 'b', 'e', 'd', 'a']]
>>> kecd = KemenyOrder(cdcd); kecd.maximalRankings
[['c', 'b', 'e', 'd', 'a']]
```

The NetFlows rule, like the Borda rule, inverts the two top ranked candidates : '*bceda*', whereas Copeland's, Kemeny's and Slater's rules result again in the same unique ranking : '*cbeda*'.

Content Lecture 3

A classification of ranking rules

Definition (Condorcet-consistency)

A ranking rule is **Condorcet-consistent** if the following holds : *If the majority relation is a linear ranking, then this ranking is the* unique *solution of the ranking rule.*

Property (Condorcet consistent rules)

Kemeny's, Slater's, Copeland's, Kohler's and the RankedPairs rule are all Condorcet-consistent. The Borda and the NetFlows rules are, both, not Condorcet-consistent.

A classification of ranking rules

Definition (*M*-ordinality)

A ranking rule is M-ordinal *if its ranking result only depends on the order of the majority margins.*

Property (*M*-ordinal rules)

Slater's, Copeland's, Kohler's and the RankedPairs rule are all M-ordinal. The Kemeny and the NetFlows rules are not M-ordinal.

Definition (*M*-invariance)

A ranking rule is M-invariant *if its ranking result only depends on the sign of the majority margins.*

Property (*M*-invariant rules)

Slater's and Copeland's rule are both M-invariant. Kohler's and the RankedPairs rules are not M-invariant.

A classification of ranking rules by Cl. Lamboray

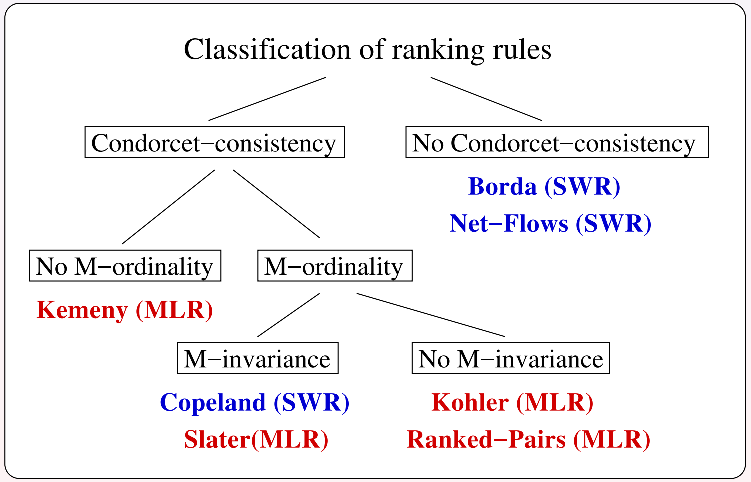


FIGURE – 4. *SWR* : single weak ranking, *MLR* : multiple linear rankings

Which ranking rule should we use

- There is no perfect ranking rule (cf Arrow's theorem).
- What properties of a ranking rule are useful or required ?
- Axiomatic characterizations of the ranking rules.
- More or less consensual global rankings ?
- Correlation with the majority margins $M(x, y)$?
- Fitness for big data : computational complexity ?

Example (Which is the *better* social ranking ?)

```
>>> from votingProfiles import *
>>> v = LinearVotingProfile('example1')
>>> v.showHTMLVotingHeatmap(rankingRule='Copeland')
>>> v.showHTMLVotingHeatmap(rankingRule='NetFlows')
```

criteria	v5	v3	v2	v4	v1
weights	2	4	7	4	8
tau(*)	0.60	0.40	0.40	0.20	0.20
c	5	4	3	2	4
b	3	3	4	5	3
e	2	2	5	3	2
d	4	5	2	4	1
a	1	1	1	1	5

criteria	v2	v5	v4	v3	v1
weights	7	2	4	4	8
tau(*)	0.60	0.40	0.40	0.20	0.00
b	4	3	5	3	3
c	3	5	2	4	4
e	5	2	3	2	2
d	2	4	4	5	1
a	1	1	1	1	5

Figure – 5. Copeland – versus *NetFlows* ranking.
(*) tau : Ordinal (Kendall) correlation between *marginal* and *global* ranking.
The ranks are of *reversed Borda type* : $w_1 = 5, w_2 = 4, w_3 = 3, w_4 = 2, w_5 = 1$.

Example (Correlations with the majority margins)

```
>>> cd.recodeValuation(1,1) # normalizing the majority margins
>>> from linearOrders import CopelandOrder, NetFlowsOrder
>>> cop = CopelandOrder(cd); cop.copelandOrder
['a', 'd', 'e', 'b', 'c']
>>> corr = cd.computeOrderCorrelation(cop.copelandOrder)
>>> cd.showCorrelation(corr)
Correlation indexes:
Crisp ordinal correlation : +1.000
Valued equivalalence      : +0.320
Epistemic determination   :  0.320
>>> nf = NetFlowsOrder(cd); nf.netFlowsOrder
['a', 'd', 'e', 'c', 'b']
>>> corr cd.computeOrderCorrelation(nf.netFlowsOrder)
Correlation indexes:
Crisp ordinal correlation : +0.925
Valued equivalalence      : +0.296
Epistemic determination   :  0.320
```

In this example, the *Condorcet-consistency* property assures that the *Copeland*, *Kemeny* and *Slater* ranking rules all deliver a perfectly matching ordinal result ($\tau = +1.0$), whereas the *Net-Flows* rule inverts the top candidates ($\tau = +0.925$). The epistemic determination of the majority margins is 0.32, ie the ordinal correlations are supported here in average by a $(1.0 + 0.32)/2 = 66\%$ majority, ie 16/25 voters.

Exercise (Claude Lamboray, PhD thesis p. 35)

Apply all the previous ranking rules on the following profile of 10 weighted linear orders defined on 4 candidates $\{a, b, c, d\}$ as shown below ; discuss the results.

4 : abcd	3 : bcad
4 : dcab	4 : dabc
4 : cabd	2 : cdab
5 : dbca	2 : bacd
1 : cbda	1 : acdb

Exercise (*votingProfiles* module extension)

Suppose that some voters will not provide a complete linear ranking of all the candidates. Develop Python code based on the `votingProfiles` module, that implements all the previously defined ranking rules and renders a corresponding ranking when given a `LinearVotingProfile` instance with partial ballots.

Digraph3 software resources

- Documentation index : https ://digraph3.readthedocs.io/en/latest/index.html
- Tutorials : https ://digraph3.readthedocs.io/en/latest/tutorial.html
- Reference manual : https ://digraph3.readthedocs.io/en/latest/techDoc.html
- Advanced topics : https ://digraph3.readthedocs.io/en/latest/pearls.html