Algorithmic Decision Theory

Lecture 2: Who wins the election ?

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Introduction

Voting is difficult

Examples (Differences and similarities)

- UK parliament member voting : 650 single-seat constituencies, one representative elected in each constituency, one vote per voter, simple majority of votes for being elected.
- Élections législatives en France : une circonscription électorale pour chaque siège, un candidat obtenant plus de 50% des voix est élu d'office, sinon il y a un deuxième tour avec tous les candidats de plus de 12.5 % des voix. Est finalement élu le candidat qui obtient le plus de voix.
- Élection présidentielle en France : chaque électeur vote pour un candidat. Si un candidat obtient plus de 50% des voix il est élu. Sinon un deuxième tour avec les deux candidats qui ont obtenu le plus de voix est organisé. Le candidat élu est alors celui qui obtient le pus de voix.

Uninominal election : definition

Principle

We suppose that each voter ranks all potential candidates from the best to the worst, without ties, and communicates this ranking without cheating.

In an uninominal election each voter votes for his best ranked candidate.

Example

- Let a,b, and c be three candidates at an election.
- Suppose that a voter prefers a to b and b to c. We simply denote this information as abc.
- In this case, the voter will vote for candidate a.

Uninominal election : properties

Example (Majority dictatorship)

- Let $\{a, b, c, \ldots, , y, z\}$ be the set of 26 candidates for a 100 voters election. Suppose that :
  - 51 voters have preferences $abc \ldots yz$, and
  - 49 voters have preferences $zbc \ldots ya$.
- 51 voters will vote for a and 49 for z.

Comment

- *In all uninominal election systems, candidate a will be elected. Is a really a good candidate ?*
- *No : Nearly half of the voters see candidate a as their worst choice ! Whereas candidate b could be an unanimous good compromise !*
- *Simple majority allows dictatorship of majority and does not favor compromise solutions.*

Uninominal election : properties

Example (Not respecting the majority of voters)

The voting system in the UK is plurality voting : The election is uninominal and the result is determined by a simple majority of votes.

- Let $\{a, b, c\}$ be the set of candidates for a 21 voters election. Suppose that :
  - 10 voters have preferences abc,
  - 6 voters have preferences bca, and
  - 5 voters have preferences cba.
- a obtains 10, b 6 and c 5 votes.

Comment

- *Candidate a is elected.*
- *This result differs a lot from what a majority of voters wants !*
- *An absolute majority of voters (11 out of 21) prefer indeed b and c over a !*

Two stage uninominal elections

Example

- Same setting as before, but we suppose this time a two stage election as in France.
- After the first stage, a obtains 10, b 6 and c 5 votes.
- Hence, no absolute majority ($> 50\%$) and there will be a second stage without candidate c.
- Suppose the voters do not change their preferences.
- a obtains eventually 10, and b 11 votes.

Comment

- *Candidate b will win the election with 15 out of 21 votes.*
- *Neither a, nor c, are preferred to b by a majority of voters this time.*
- *Is this two stage voting system therefore always more satisfactory ?*

Two stage uninominal elections : properties

Example (Not-respecting the majority of voters)

- Let $\{a, b, c, d\}$ be the set of candidates for a 21 voters election. Suppose that :
  - 10 voters have preferences $bacd$,
  - 6 voters have preferences $cadb$, and
  - 5 voters have preferences $adbc$.
- At the first stage : b obtains 10, c 6 and a 5 votes.
- Their will be a second stage election with candidates $\{b, c\}$.
- This time b obtains 15, and c 6 votes.
- Candidate b is consequently elected.

Two stage uninominal elections : properties

Example (Not-respecting the majority of voters – continue)

- The previous result is clearly different from what a majority of voters prefer :
- Remind that :
  - 10 voters have preferences *bacd*,
  - 6 voters have preferences *cadb*, and
  - 5 voters have preferences *adbc*.
- Indeed, an absolute majority (11 out of 21) apparently prefers *a* and *d* over *b* !

Two stage uninominal elections : properties

Example (Manipulation in two-stage uninominal elections)

- Same setting as before, but we suppose that the 6 voters who have previously voted in favor of *c* are going to cheat, and vote instead for *a*, their second best choice.
- In this case, *a* obtains 11, and *b* 10 votes.

Comment

- *Thus, candidate a is elected with absolute majority right at the first stage.*
- *By cheating, these voters obtain a better result than if they were voting following their preferences.*
- *An election system which favors this kind of strategic (cheating) votings is called manipulable.*
- *This is not the only weakness of the French voting system.*

Two-stage uninominal elections : properties

Example (monotonicity violation in the two-stage voting system)

- Let $\{a, b, c\}$ be the set of candidates for a 17 voters election. Suppose that a pre-election survey reveals that :
  - 6 voters will have preferences *abc*,
  - 5 voters will have preferences *cab*,
  - 4 voters will have preferences *bca*, and
  - 2 voters will have preferences *bac*.
- After the first stage : *a* will obtain 6, *b* 6 and *c* 5 votes. There probably will be a second stage with running candidates $\{a, b\}$.
- This time, *a* will obtain 11, and *b* 6 votes ; consequently *a* will be elected with a comfortable majority.

Violating monotonicity of apparent preferences

Example (Continue)

- Suppose that, following the survey, candidate *a* wants to increase his lead over *b* and strengthens his election campaign against *b*. Suppose that he succeeds in winning the two last voters for him (preferences *bac* become *abc*).
- After the first stage, it is now candidate *b* who is eliminated from the second stage.
- The 2nd stage opposes this time *a* and *c* and it will be candidate *c* who eventually wins this election with a large majority.

Two-stage uninominal elections : properties

Example (Favoring strategic abstentions)

- Let $\{a, b, c\}$ be the set of candidates for a 11 voters election. Suppose that :
  - 4 voters have preferences *abc*,
  - 4 voters have preferences *cba*, and
  - 3 voters have preferences *bca*.
- There will be a second stage election opposing candidates $\{a, c\}$ and evetually c will be elected.
- As c is their worst candidates, 2 out of 4 voters from the first group decide not to go for voting in the first stage.

Exercise(s)

Show that this strategic abstention is profitable for these two voters.

Two-stage uninominal election : properties

Example (Manipulation by constituency configuration)

- Let $\{a, b, c\}$ be the set of candidates for a 26 voters election divided into two constituencies : the town (13) and the countryside (13). Suppose that the 13 voters in town have the following preferences :
  - 4 voters have preferences *abc*,
  - 3 voters have preferences *bac*,
  - 3 voters have preferences *cab*, and
  - 3 voters have preferences *cba*.
- Suppose that the 13 voters in the countryside show the following preferences :
  - 4 voters have preferences *abc*,
  - 3 voters have preferences *cab*,
  - 3 voters have preferences *bca*, and
  - 3 voters have preferences *bac*.

Exercise(s)

Show that, when joining the two constituencies into a single one, the election result will be different from the one obtained with the two constituencies.

Sequential pairwise elections : properties

Example (Influence of the agenda)

- Let $\{a, b, c\}$ be the set of candidates for a 3 voters election. Suppose that :
  - 1 voter has preferences *abc*,
  - 1 voter has preferences *bca*, and
  - 1 voter has preferences *cab*.
- The candidates will be considered two by two along the following agenda : a and b first, than c.
- In the first vote, a is opposed to b and wins the election (2 votes against 1).
- Then a is opposed to c and c eventually wins with 2 votes against 1.

Exercise(s)

What happens if the agenda is : a and c first ? What if b and c come first ?

Lack of neutrality in sequential elections

Comment

- *In this example, any candidate can be elected, it only depends on the agenda. A sequential election system always lacks neutrality of the candidates.*
- *Note that sequential voting is very frequent in parliaments, where the amendements to a bill are considered in a predefined sequence.*
- *The same happens often in administration and gouverning boards, which gives the presidents of these decision bodies an effective manipulation power.*

Sequential pairwise elections : properties

Example (Violation of unanimity)

- Let $\{a, b, c, d\}$ be the set of candidates for a 3 voters election. Suppose that :
  - 1 voter has preferences *badc*,
  - 1 voter has preferences *cbad*, and
  - 1 voter has preferences *adcb*.
- Consider the following agenda : a and b first, than c, and finally d.
- Candidate a is defeated by b in the first round. Candidate c wins then the second round, and d eventually wins the election.
- Notice that all voters unanimously prefer candidate a over d ! ? !.

Comment

This can evidently not happen with uninominal election systems whether two-stage or not.

Finding the winner by aggregating marginal rankings

1. Each voter ranks again without ties the potential candidates from his best to his worst candidate and communicates without cheating this ranking.
2. The election result is computed by aggregating directly these marginal rankings into a global compromise one.

Comment

Two seminal aggregation methods, quite different in their spirit, have been proposed in the 18th century by two French scientists :

> *Marie Jean Antoine Nicolas Caritat, marquis de Condorcet (17 septembre 1743 – 28 mars 1794) mathématicien, philosophe et politologue.*
>
> *Jean-Charles Chevalier de Borda (4 mai 1733 – 19 février 1799) ingénieur du génie maritime, mathématicien, physicien et politologue.*

Condorcet's method

Principle (Condorcet 18th century)

- In 1785, Condorcet suggests to compare pairwise all the potential candidates.
- Candidate a is preferred to candidate b if and only if the number of voters who rank a before b is higher than the number of voters who ranks b before a.
- A candidate, who is thus preferred to all the others, wins the election and is called Condorcet winner.

Condorcet's method

Comment

- *The Condorcet winner is always preferred by a majority of voters to all the other candidates.*
- *He always defeats all the other candidates in a sequential election.*
- *A Condorcet winner is always unique.*

Condorcet's

Example (The Condorcet winner)

- Let $\{a, b, c, d, e, f, g, x, y\}$ be the set of candidates for a 101 voters election. Suppose that :
  - 19 voters have preferences $yabcdefgx$,
  - 21 voters have preferences $efgxyabcd$,
  - 10 voters have preferences $exyabcdfg$,
  - 10 voters have preferences $fxyabcdeg$,
  - 10 voters have preferences $gxyabcdef$,and
  - 31 voters have preferences $yabcdxefg$.
- Candidate x is here the Condorcet winner.

Exercise(s)

Write a Python program for computing the Condorcet winner when given the results of an n voters election with p candidates.

Weaknesses of Condorcet's method

Comment

- *Let us compare the election results for candidates x and y by counting the voters who have ranked these candidates at rank $k = 1$ to 9.*

	k=1	2	3	4	5	6	7	8	9
x	0	30	0	21	0	31	0	0	19
y	50	0	30	0	21	0	0	0	0

1. *Candidate y seams to be globally much better appreciated than the Condorcet winner x !*
2. *There may not exist a Condorcet winner !*

Exercise(s)

Find an example of election where there is no Condorcet winner.

Borda's method

Contemporary with Condorcet, Borda invented his nowadays famous scoring method for computing the winner of an election :

Principle (Borda, 18th century)

In the marginal ordering of each voter, every candidate appears at a certain rank : 1 fort the first, 2 for second, etc.

The sum of marginal ranks obtained by each candidate is called its Borda score.

A candidate showing the smallest Borda score wins the election and is called Borda winner.

Borda's method

Comment

- *A Borda winner might not be unique. In this case all Borda winners are considered equally preferred.*
- *Borda's methods, besides determining the Borda winner(s), renders by the way a complete preorder (ranking with possible ties) of the candidates.*

Exercise(s)

Write a Python program for computing the Borda winner and ranking the candidates when given the results of an n voters election with p candidates.

Comparing Condorcet's with Borda's method

Example

- Let $\{a, b, c, d\}$ be the set of candidates for a 3 voters election. Suppose that :
  - 2 voters have preferences *bacd*, and
  - 1 voter has preferences *acdb*.
- The Borda score of a is $2 \times 2 + 1 \times 1 = 5$
- The Borda score of b est $2 \times 1 + 1 \times 4 = 6$
- The Borda score of c est $2 \times 3 + 1 \times 2 = 8$
- The Borda score of d est $2 \times 4 + 1 \times 3 = 11$

Comment

- *Borda winner is a.*
- *Condorcet winner is b.*

Comparing Condorcet's with Borda's method

Example (Independance of irrelevant alternatives (IIA))

- Let $\{a, b, c\}$ be the set of candidates for a 2 voters election. Suppose that :
  - 1 voter has preferences *acb*, and
  - 1 voter has preferences *bac*.
- The Borda winner is a.
- Consider now a different appreciation of candidate c :
  - 1 voter has preferences *abc*, and
  - 1 voter has preferences *bca*.
- Now, the Borda winner is b.

Comment

Condorcet's method, being pairwise, naturally verifies this property !

Some theoretical results

Comment

- *universality : The election system must be applicable to all possible voting outcomes.*
- *transitivity : The outcome must be a transitive ordering, possibly with ties.*
- *unanimity : If all voters rank candidate a before candidate b then a must also be ranked before b in the global compromise ranking.*
- *independence (IIA) : – The difference in the global ranks of two candidates only depends on their respective marginal ranks.*
- *non-dictatorship : No voter may systematically impose his ordering as the global one.*

Some theoretical results

Theorem (K. Arrow, 1963)

If the number of candidates is at least three, there is no aggregation method of marginal rankings that can satisfy at the same time : universality, unanimity, transitivity, independence and non-dictatorship.

Comment

- *Borda's method verifies : universality, unanimity, transitivity, and non-dictatorship.*
- *Condorcet's method, verifies : universality, unanimity, independence, and non-dictatorship.*

Some theoretical results

Theorem (Gibbard & Satterthwaite, 1973 et 1975)

If the number of candidates in an election is at least two, there is no marginal rankings aggregation method that can verify at the same time : universality, non-dictatorship, and non-manipulability.

Comment

- *The French and British election systems verify universality and non-dictatorship.*
- *Therefore they are inevitably manipulable.*

Complexity of determining winner

- How quickly can we determine the result under a certain voting rule ?
  - n candidates ; v voters.
  - Plurality : $O(n)$
  - Condorcet and Borda winners : $O(nv)$
- Even low order polynomials would be a problem in real elections : U.S. presidential elections with an $O(v^3)$ algorithm ?
- Can it get worse ?

Complexity of determining winner – continue

- **Dodgson's (lewis Caroll) rule** : The winner of an election is the candidate who requires the fewest preference switches (adjacent) to become the winner.
- **Theorem** (Bartholdi, Tovey, and Trick, BTT 1989) :

  It is NP-hard to determine the winner of an election under Dodgson's Method.
- **Kemeny's Rule :** Find an ordering that is "closest" to the voters' preferences (so if a beats b by 3 votes, then it costs 3 to reverse this).

  Kemeny's rule is also NP-hard.

Complexity of determining winner – continue

- **Definition.** A voting system satisfies neutrality if it is symmetric in its treatment of the candidates.
- **Definition.** A voting system satisfies consistency if, when two disjoint sets of voters agree on a candidate c, the union of voeters will also choose c.
- **Impracticality Theorem (BTT, 1989).**

  For any voting system that satisfies
  (a) neutrality
  (b) consistency
  (c) Condorcet winner

  it is NP-Hard to determine the winner.

Electing the Doge of Venice

1. Thirty members of the Great Council, chosen by lot, were reduced by lot to nine.
2. The nine chose forty and the forty were reduced by lot to twelve, who chose twenty-five.
3. The twenty-five were reduced by lot to nine and the nine elected forty-five.
4. Then the forty- five were once more reduced by lot to eleven.
5. And the eleven finally chose the forty-one,
6. who actually elected the doge.

Complexity of manipulation

- Can it ever be hard to manipulate ?

  Yes
- **Definition.** The Copeland score of a candidate is the number of pairwise contests won minus the number lost.
- **Definition.** The Second order Copeland score of a candidate is the sum of the Copeland scores of each defeated candidate.
- **Theorem** (BTT 1989).
  It is NP-complete for a voter to determine how to manipulate an election under second order Copeland score.
- There are others.

  Single Tranferable Vote – Instant Runoff Voting – is the most natural system (Bartholdi and Orlin).

Manipulation by Chairs

- Chairs of committees may have a number of powers :
- Changing the Candidates
  1. Adding Candidates
  2. Deleting Candidates
  3. Partitioning Candidates
- Changing the Voters
  1. Adding Voters
  2. Deleting Voters
  3. Partitioning Voters
- Many “*fairness conditions*” address the question of whether a voting rule is vulnerable to these sort of manipulations.